

Compactness of Hankel Operators with Conjugate Holomorphic Symbols on Complete Reinhardt Domains in \mathbb{C}^2

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Abstract

In this paper we characterize compact Hankel operators with conjugate holomorphic symbols on the Bergman space of bounded convex Reinhardt domains in \mathbb{C}^2 . We also characterize compactness of Hankel operators with conjugate holomorphic symbols on smooth bounded pseudoconvex complete Reinhardt domains in \mathbb{C}^2 .

1 Introduction

We assume $\Omega \subset \mathbb{C}^2$ is a bounded convex Reinhardt domain. We denote the Bergman space with the standard Lebesgue measure on Ω as $A^2(\Omega)$. Recall that the Bergman space $A^2(\Omega)$ is the space of holomorphic functions on Ω that are square integrable on Ω under the standard Lebesgue measure. The Bergman space is a closed subspace of $L^2(\Omega)$. Therefore there exists an orthogonal projection $P : L^2(\Omega) \rightarrow A^2(\Omega)$ called the Bergman projection. The Hankel operator with symbol ϕ is defined as $H_\phi g = (I - P)(\phi g)$ for all $g \in A^2(\Omega)$. If $\phi \in L^\infty(\Omega)$, then H_ϕ is a bounded operator, however, the converse is not necessarily true. For example, let $h \in A^2(\Omega) \setminus L^\infty(\Omega)$ then H_h is a densely defined operator (since any holomorphic function that is smooth up to the boundary is in the domain of H_h). Furthermore, $H_h = 0$ on this dense set and so extends continuously to all of $A^2(\Omega)$.

Let $h \in A^2(\Omega)$ so that the Hankel operator $H_{\bar{h}}$ is compact on $A^2(\Omega)$. The Hankel operator with an $L^2(\Omega)$ symbol may only be densely defined, since the product of L^2 functions may not be in L^2 . However, if compactness of the Hankel operator is also assumed, then the Hankel operator with an L^2 symbol is defined on all of $A^2(\Omega)$.

We wish to use the geometry of the boundary of Ω to give conditions on h . For example, if Ω is the bidisk, Le in [5, Corollary 1] shows that if $h \in A^2(\mathbb{D}^2)$ such that $H_{\bar{h}}$ is compact on $A^2(\mathbb{D}^2)$ then $h \equiv c$ for some $c \in \mathbb{C}$. In one variable, Axler in [1] showed that $H_{\bar{g}}$ is compact on $A^2(\mathbb{D})$ if and only if g is in the little Bloch space. That is, $\lim_{|z| \rightarrow 1^-} (1 - |z|^2)g'(z) = 0$. If the symbol h is smooth up to the boundary of a smooth bounded convex domain in \mathbb{C}^2 , Čučković and Şahutoğlu in [2] showed that Hankel operator H_h is compact if and only if h is holomorphic along analytic disks in the boundary of the domain.

In this paper we will use the following notation.

$$S_t = \{z \in \mathbb{C} : |z| = t\},$$

$$\mathbb{T}^2 = S_1 \times S_1 = \{z \in \mathbb{C} : |z| = 1\} \times \{w \in \mathbb{C} : |w| = 1\},$$

$$\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$$

for any $r, t > 0$. If $r = 1$ we write

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

We say $\Delta \subset b\Omega$ is an analytic disk if there exists a function $h = (h_1, h_2) : \mathbb{D} \rightarrow b\Omega$ so that each component function is holomorphic on \mathbb{D} and the image $h(\mathbb{D}) = \Delta$. An analytic disk is said to be trivial if it is degenerate (that is, $\Delta = (c_1, c_2)$ for some constants c_1 and c_2).

In [4] we considered bounded convex Reinhardt domains in \mathbb{C}^2 . We characterized non trivial analytic disks in the boundary of such domains.

We defined

$$\Gamma_\Omega = \overline{\bigcup \{\phi(\mathbb{D}) | \phi : \mathbb{D} \rightarrow b\Omega \text{ are holomorphic, non-trivial}\}}$$

and showed that

$$\Gamma_\Omega = \Gamma_1 \cup \Gamma_2$$

where either $\Gamma_1 = \emptyset$ or

$$\Gamma_1 = \overline{\mathbb{D}_{r_1}} \times S_{s_1}$$

and likewise either $\Gamma_2 = \emptyset$ or

$$\Gamma_2 = S_{s_2} \times \overline{\mathbb{D}_{r_2}}$$

for some $r_1, r_2, s_1, s_2 > 0$.

The main results are the following theorems.

Theorem 1. Let $\Omega \subset \mathbb{C}^2$ be a bounded convex Reinhardt domain. Let $f \in A^2(\Omega)$ so that $H_{\bar{f}}$ is compact on $A^2(\Omega)$. If $\Gamma_1 \neq \emptyset$, then f is a function of z_2 alone. If $\Gamma_2 \neq \emptyset$, then f is a function of z_1 alone.

Corollary 1. Let $\Omega \subset \mathbb{C}^2$ be a bounded convex Reinhardt domain. Suppose $\Gamma_1 \neq \emptyset$ and $\Gamma_2 \neq \emptyset$. Let $f \in A^2(\Omega)$ so that $H_{\bar{f}}$ is compact on $A^2(\Omega)$. Then there exists $c \in \mathbb{C}$ so that $f \equiv c$.

Theorem 2. Let $\Omega \subset \mathbb{C}^2$ be a C^∞ -smooth bounded pseudoconvex complete Reinhardt domain. Let $f \in A^2(\Omega)$ such that $H_{\bar{f}}$ is compact on $A^2(\Omega)$. If either of the following conditions hold:

1. There exists a holomorphic function $F = (F_1, F_2) : \mathbb{D} \rightarrow b\Omega$ so that both F_1 and F_2 are not identically constant.
2. $\Gamma_1 \neq \emptyset$ and $\Gamma_2 \neq \emptyset$

Then $f \equiv c$ for some $c \in \mathbb{C}$.

2 Preliminary Lemmas

As a bit of notation to simplify the reading, we will use the multi-index notation. That is, we will write

$$z = (z_1, z_2)$$

and

$$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2}$$

and $|\alpha| = \alpha_1 + \alpha_2$. We say $\alpha = \beta$ if $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$. If either $\alpha_1 \neq \beta_1$ or $\alpha_2 \neq \beta_2$ we say $\alpha \neq \beta$.

It is well known that for bounded Reinhardt domains in \mathbb{C}^2 , the monomials

$$\left\{ \frac{z^\alpha}{\|z^\alpha\|_{L^2(\Omega)}} : \alpha \in \mathbb{Z}_+^2 \right\}$$

form an orthonormal basis for $A^2(\Omega)$.

We denote

$$\frac{z^\alpha}{\|z^\alpha\|_{L^2(\Omega)}} = e_\alpha(z)$$

Definition 1. For $\beta = (\beta_1, \beta_2) \in \mathbb{Z}^2$, we define

$$G_\beta := \left\{ \psi \in L^2(\Omega) : \psi(\zeta z) = \zeta^\beta \psi(z) \text{ a.e. } z \in \Omega \text{ a.e. } \zeta \in \mathbb{T}^2 \right\}$$

Note this definition makes sense in the case Ω is a Reinhardt domain, and is the same as the definition of quasi-homogeneous functions in [5].

Lemma 1. *Let $\Omega \subset \mathbb{C}^2$ be a bounded complete Reinhardt domain. G_α as defined above are closed subspaces of $L^2(\Omega)$ and for $\alpha \neq \beta$,*

$$G_\alpha \perp G_\beta$$

Proof. The proof that G_β is a closed subspace of $L^2(\Omega)$ is similar to [5]. Without loss of generality, suppose $\alpha_1 \neq \beta_1$. Since Ω is a complete Reinhardt domain, one can 'slice' the domain similarly to [3]. That is,

$$\Omega = \bigcup_{z_2 \in H_\Omega} (\Delta_{|z_2|} \times \{z_2\})$$

where $H_\Omega \subset \mathbb{C}$ is a disk centered at 0 and

$$\Delta_{|z_2|} = \{z \in \mathbb{C} : |z| < r_{|z_2|}\}$$

is a disk with radius depending on $|z_2|$. As we shall see, the proof relies on the radial symmetry of both H_Ω and $\Delta_{|z_2|}$.

Let $f \in G_\alpha$ and $g \in G_\beta$ and $z_1 = r_1 \zeta_1$, $z_2 = r_2 \zeta_2$ for $(\zeta_1, \zeta_2) \in \mathbb{T}^2$ and $r_1, r_2 \geq 0$ then we have

$$\begin{aligned} & \langle f, g \rangle \\ &= \int_{\Omega} f(z) \overline{g(z)} dV(z) \\ &= \int_{H_\Omega} \int_{0 \leq r_1 \leq r_{|z_2|}} \int_{\mathbb{T}} \zeta_1^{\alpha_1} \overline{\zeta_1}^{\beta_1} f(r_1, z_2) \overline{g(r_1, z_2)} r_1 d\sigma(\zeta_1) dr_1 dV(w). \end{aligned}$$

Since $\alpha_1 \neq \beta_1$,

$$\int_{\mathbb{T}} \zeta_1^{\alpha_1} \overline{\zeta_1}^{\beta_1} d\sigma(\zeta_1) = 0.$$

This completes the proof. □

In the case of a bounded convex Reinhardt domain in \mathbb{C}^2 , one can use the 'slicing' approach in [3] to explicitly compute $P(\bar{z}^j e_n)$.

Lemma 2. *Let $\Omega \subset \mathbb{C}^2$ be a bounded complete Reinhardt domain. Then the Hankel operator with*

symbol $\bar{z}^j \bar{w}^k$ applied to the orthonormal basis vector e_n has the following form:

$$H_{\bar{z}^j} e_n(z) = \frac{\bar{z}^j z^n}{\|z^n\|}$$

if either $n_1 - j_1 < 0$ or $n_2 - j_2 < 0$. If $n_1 - j_1 \geq 0$ and $n_2 - j_2 \geq 0$ then we can express the Hankel operator applied to the standard orthonormal basis as

$$H_{\bar{z}^j} e_n(z) = \frac{\bar{z}^j z^n}{\|z^n\|} - \frac{z^{n-j} \|z^n\|}{\|z^{n-j}\|^2}.$$

Furthermore, for any monomial

$$\bar{w}^j w^n \in G_{n-j}$$

the projection

$$(I - P)(\bar{w}^j w^n) \in G_{n-j}.$$

Proof. We have

$$\begin{aligned} P(\bar{z}^j e_n)(z) &= \\ &= \int_{\Omega} \bar{w}^j \frac{w^n}{\|w^n\|} \sum_{l \in \mathbb{Z}_+^2} \overline{e_l(w)} e_l(z) dV(z, w) \\ &= \int_{H_{\Omega}} \int_{w_1 \in \Delta_{|w_2|}} \bar{w}_1^{j_1} \bar{w}_2^{j_2} \frac{w_1^{n_1} w_2^{n_2}}{\|z^n\|} \sum_{l_1, l_2=0}^{\infty} \overline{e_{l_1, l_2}(w_1, w_2)} e_{l_1, l_2}(z_1, z_2) dA_1(w_1) dA_2(w_2) \\ &= \sum_{l_1, l_2=0}^{\infty} \frac{z_1^{l_1} z_2^{l_2}}{\|z^n\| \|z^l\|^2} \int_{H_{\Omega}} \bar{w}_2^{j_2 + l_2} w_2^{n_2} \int_{w_1 \in \Delta_{|w_2|}} \bar{w}_1^{j_1 + l_1} w_1^{n_1} dA_1(w_1) dA_2(w_2). \end{aligned}$$

Converting to polar coordinates and using the orthogonality of $\{e^{in\theta} : n \in \mathbb{Z}\}$ and the fact that

$$\int_{w_1 \in \Delta_{|w_2|}} \bar{w}_1^{j_1 + l_1} w_1^{n_1} dA_1(w_1)$$

is a radial function of w_2 and H_{Ω} is radially symmetric, we have the only non-zero term in the previous sum is when $n_2 - j_2 = l_2$ and $n_1 - j_1 = l_1$. Therefore, we have $P(\bar{w}^j e_n)(z) = 0$ if $n_2 - j_2 < 0$ or $n_1 - j_1 < 0$. Otherwise, if $n_2 - j_2 \geq 0$ and $n_1 - j_1 \geq 0$, we have

$$P(\bar{w}^j e_n)(z) = \frac{z^{n-j} \|z^n\|}{\|z^{n-j}\|^2}.$$

Therefore, we have

$$H_{\bar{w}^j} e_n(z) = \frac{\bar{z}^j z^n}{\|z^n\|} - \frac{z^{n-j} \|z^n\|}{\|z^{n-j}\|^2}$$

if $n_2 - k \geq 0$ and $n_1 - j \geq 0$ otherwise

$$H_{\bar{w}^j} e_n(z) = \frac{\bar{z}^j z^n}{\|z^n\|}$$

if either $n_2 - k < 0$ or $n_1 - j < 0$. This also shows that the subspaces G_α remain invariant under the projection $(I - P)$, at least for monomial symbols. \square

Lemma 3. *For every $\alpha \geq 0$, the product Hankel operator*

$$H_{\bar{z}^\alpha}^* H_{\bar{z}^\alpha} : A^2(\Omega) \rightarrow A^2(\Omega)$$

is a diagonal operator with respect to the standard orthonormal basis

$$\{e_j : j \in \mathbb{Z}_+^2\}.$$

Proof. Assume without loss of generality, $j \neq l$. We have

$$\begin{aligned} & \langle H_{\bar{z}^\alpha}^* H_{\bar{z}^\alpha} e_j, e_l \rangle \\ &= \langle H_{\bar{z}^\alpha} e_j, H_{\bar{z}^\alpha} e_l \rangle \\ &= \langle (I - P)(\bar{z}^\alpha e_j), \bar{z}^\alpha e_l \rangle. \end{aligned}$$

We have $\bar{z}^\alpha e_j \in G_{j-\alpha}$, $\bar{z}^\alpha e_l \in G_{l-\alpha}$. By Lemma 2,

$$(I - P)\bar{z}^\alpha e_j \in G_{j-\alpha}.$$

By Lemma 1, G_α are mutually orthogonal. Therefore,

$$\langle (I - P)(\bar{z}^\alpha e_j), \bar{z}^\alpha e_l \rangle = 0$$

unless $j = l$. \square

Using Lemma 2 and Lemma 3, let us compute the eigenvalues of

$$H_{\bar{z}^\alpha}^* H_{\bar{z}^\alpha}.$$

Let us first assume $n - \alpha \geq 0$. We have

$$\begin{aligned}\langle H_{\bar{z}^\alpha}^* H_{\bar{z}^\alpha} e_n, e_n \rangle &= \left\langle \frac{\bar{z}^\alpha z^n}{\|z^n\|} - \frac{z^{n-\alpha} \|z^n\|}{\|z^{n-\alpha}\|^2}, \frac{\bar{z}^\alpha z^n}{\|z^n\|} \right\rangle \\ &= \frac{\|\bar{z}^\alpha z^n\|^2}{\|z^n\|^2} - \frac{\|z^n\|^2}{\|z^{n-\alpha}\|^2}.\end{aligned}$$

If $n - \alpha < 0$, we have

$$\langle H_{\bar{z}^\alpha}^* H_{\bar{z}^\alpha} e_n, e_n \rangle = \frac{\|\bar{z}^\alpha z^n\|^2}{\|z^n\|^2}.$$

3 Proof of Theorem 1

Proof. Assume $f \in A^2(\Omega)$ and $H_{\bar{f}}$ is compact on $A^2(\Omega)$. Then, we can represent

$$f = \sum_{j,k=0}^{\infty} c_{j,k,f} z_1^j z_2^k$$

almost everywhere (with respect to the Lebesgue volume measure on Ω). Let

$$\{e_m : m \in \mathbb{Z}_+^2\}$$

be the standard orthonormal basis for $A^2(\Omega)$. Then

$$\|H_{\bar{f}} e_m\|^2 \rightarrow 0$$

as $|m| \rightarrow \infty$. Using the mutual orthogonality of the subspaces $G_{\alpha,\beta}$, we get

$$\begin{aligned}\|H_{\bar{f}} e_m\|^2 &= \langle (I - P)(\bar{f} e_m), \bar{f} e_m \rangle \\ &= \left\langle \sum_{j,k=0}^{\infty} (I - P)(\overline{c_{j,k,f}} z_1^j \bar{z}_2^k e_m), \sum_{s,p=0}^{\infty} \overline{c_{s,p,f}} z_1^s \bar{z}_2^p e_m \right\rangle \\ &= \sum_{j,k=0}^{\infty} \|H_{\overline{c_{j,k,f}} z_1^j \bar{z}_2^k} e_m\|^2 \geq \|H_{\overline{c_{j,k,f}} z_1^j \bar{z}_2^k} e_m\|^2\end{aligned}$$

for every $(j, k) \in \mathbb{Z}_+^2$. Taking limits as $|m| \rightarrow \infty$, we have $\lim_{|m| \rightarrow \infty} \|H_{c_{j,k,f}z_1^j z_2^k} e_m\|^2 = 0$ for all $(j, k) \in \mathbb{Z}_+^2$. Since the Hankel operators

$$H_{c_{j,k,f}z_1^j z_2^k}^* H_{c_{j,k,f}z_1^j z_2^k}$$

are diagonal by Lemma 3, with eigenvalues

$$\lambda_{j,k,m} = \|H_{c_{j,k,f}z_1^j z_2^k} e_m\|^2.$$

This shows that

$$H_{c_{j,k,f}z_1^j z_2^k}^* H_{c_{j,k,f}z_1^j z_2^k}$$

are compact for every $(j, k) \in \mathbb{Z}_+^2$. Then $H_{c_{j,k,f}z_1^j z_2^k}$ are compact on $A^2(\Omega)$.

Without loss of generality, assume $\Gamma_1 \neq \emptyset$. Then there exists a holomorphic function $F = (F_1, F_2) : \mathbb{D} \rightarrow b\Omega$ so that F_2 is identically constant and F_1 is non-constant. Therefore, by [4], the composition

$$\overline{c_{j,k,f}F_1(z)^j F_2(z)^k}$$

must be holomorphic in z . This cannot occur unless $\overline{c_{j,k,f}} = 0$ for $j > 0$. Therefore, using the representation

$$f = \sum_{j,k=0}^{\infty} c_{j,k,f} z_1^j z_2^k$$

we have $f = \sum_{k=0}^{\infty} c_{0,k,f} z_2^k$ almost everywhere. By holomorphicity of f and the identity principle, this implies

$$f \equiv \sum_{k=0}^{\infty} c_{0,k,f} z_2^k.$$

Hence f is a function of only z_2 . The proof is similar if $\Gamma_2 \neq \emptyset$. □

3.1 Proof of Theorem 2

Using the same argument in the proof of Theorem 1, one can show compactness of $H_{\vec{f}}$ implies compactness of $H_{c_{j,k,f}z_1^j z_2^k}$ for every $j, k \in \mathbb{Z}_+$. Hence by [2, Corollary 1], for any holomorphic function $\phi = (\phi_1, \phi_2) : \mathbb{D} \rightarrow b\Omega$, we have

$$\overline{c_{j,k,f}\phi_1^j \phi_2^k}$$

must be holomorphic. If we assume condition two in Theorem 2, then it follows that $f \equiv c_{0,0,f}$. Assuming condition one in Theorem 2, we may assume ϕ_1 and ϕ_2 are not identically constant. Thus $c_{j,k,f} = 0$ for $j > 0$ or $k > 0$ and so $f \equiv c_{0,0,f}$.

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